MOMENTS ABOUT SMALLEST SAMPLE VALUE

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FOREWORD

This report was prepared by Prof. Dr. Wallodi Weibull,
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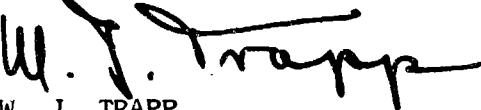
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ABSTRACT

A new type of moments has been achieved by substituting in
the central moments the smallest value of the sample for its mean.
The new moments have the same advantage as the central moments of
being independent of the location parameter but for certain values
of the shape parameter they have less variance and thus are pre-
ferable for estimating purposes. The asymptotic properties of four
estimators, three of them composed of the new moments and one of
them of central moments have been examined. It could be concluded
that for the shape parameter $\alpha \geq 0.5$ the estimator, which was composed
of the first and second order moments of the new type, was by far the
most efficient one. Small-sample properties of the new-moments
estimators have been appraised by use of extensive Monte-Carlo
studies and it could be stated that the same conclusion applies also
to small and moderate sample sizes.

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Moments about Smallest Sample Value

1. Introduction

The central moments, defined by

$$m_v = \Sigma (x_i - \bar{x})^v / n \tag{1}$$

where $\bar{x}$ denotes the mean of the sample values, has the useful property of being independent of the location parameter as easily seen by introducing the standardized variate

$$z = (x - \mu)/\beta$$

or

$$x = \beta z + \mu \tag{2}$$

Thus

$$m_v = \beta^v \Sigma (z_i - \bar{z})^v / n = \beta^v . f_v(\alpha)/n \tag{3}$$

is uniquely determined by α and β.

Considering that the variances of the mean $\bar{x}$ and of the smallest sample value x_1 from a Weibull population are

$$Var(\bar{x}) = \mu_2/n \quad \text{and} \quad Var(x_1) = \mu_2/n^{2\alpha} \tag{4}$$

it can be concluded that for $\alpha > 0.5$ the statistic x_1 is much more precise than $\bar{x}$ for large sample sizes n. Thus it seems plausible that the efficiency of the moment estimators can be increased by substituting x_1 for $\bar{x}$ in equ.(1).

In this way a new type of moments is obtained, which will be called "moments about smallest sample value", be denoted by r_v and defined by

$$r_v = \Sigma(x_i - x_1)^v / n \tag{5}$$

The properties of these moments and their usefulness for estimating purposes will now be examined.

2. Properties of the new moments

Introducing (2) into (5) we have

$$r_v = \beta^v \Sigma(z_i - z_1)^v / n = \beta^v f_{rv} (\alpha)/n \tag{6}$$

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from which it can be concluded, since z is uniquely determined by α, that the moments r_ν, just like the moments m_ν, are independent of the location parameter μ.

It should be noted that, by definition, the first central moment

$$m_1 = \Sigma(x_i - \bar{x})/n = 0 \tag{7}$$

whereas the moment

$$r_1 = \Sigma(x_i - x_1)/n = \bar{x} - x_1 > 0 \tag{8}$$

and thus offers a new tool for estimating purposes.

From (8) it follows that the expected value of r_1

$$E\,r_1 = E\bar{x} - E\,x_1 \tag{9}$$

and considering that

$$E\bar{x} = \mu + \beta\,g_1 \quad \text{and} \quad E\,x_1 = \mu + \beta\,g_1/n^\alpha \tag{10}$$

where, using a notation proposed by Dubey,

$$g_1 = \alpha\,! \quad \text{and} \quad g_\nu = (\nu\alpha)! \tag{11}$$

the expected value of r_1 becomes

$$E\,r_1 = \beta\,g_1(1 - n^{-\alpha}) \tag{12}$$

Further, from equ. (8)

$$\text{Var}(r_1) = \text{Var}(\bar{x}) + \text{Var}(x_1) + 2\,\text{Cov}\,(\bar{x}, x_1) \tag{13}$$

The first two terms are given by equ. (4), while the third term includes terms of the type $E(x_j x_1)$, which can be computed by means of formulae derived by Lieblein [1].

In the same way, the expected value of $E\,r_2$ of the second moment

$$r_2 = \Sigma\,(x_i - x_1)^2/n = a_2 - 2\,x_1 \cdot a_1 + x_1^2 \tag{14}$$

can be evaluated.

The expected values and the variances of the moments r_ν

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take very simple forms for large sample sizes, since

$$r_v = \Sigma(x_i - x_1)^v/n = \beta^v \Sigma(z_i - z_1)^v/n \longrightarrow \beta^v \Sigma z_i^v/n \tag{15}$$

as $n \longrightarrow \infty$.

From formulae given in Scientific Report No.6 (SR 6) it then follows that

$$Er_v = \beta^v \cdot g_v \tag{16}$$

and

$$n.Var(r_v) = \beta^{2v}(g_{2v} - g_v^2) \quad \text{and} \quad n.Cov(r_v, r_w) = \beta^{v+w}(g_{v+w} - g_v \cdot g_w) \tag{17}$$

3. Use of the moments for estimating shape parameters

From equ.(6) it is easy to see that by use of two moments r_v of different orders, an expression can be constructed that is independent of β and thus a function of α only, for example, the expressions

$$R_{12} = r_2/r_1^2 \qquad R_{13} = r_3/r_1^3 \qquad R_{23} = r_3^2/r_2^3 \tag{18}$$

Each such expression can be used as an estimator of α.

Since $m_1 = 0$, the only corresponding estimator, composed of central moments, is

$$M_{23} = m_3/m_2^{3/2} \tag{19}$$

The efficiency of these estimators will now be examined. Due to the fact that the moments r_v for large sample sizes tend to the moments about origin of the standardised variate, closed expressions can in this case be deduced. For small and moderate sample sizes the Monte-Carlo Method has to be used.

4. Asymptotic properties of various estimators

If $n \longrightarrow \infty$, the estimators R tend to

$$R_{12} = a_2/a_1^2 \qquad R_{13} = a_3/a_1^3 \qquad R_{23} = a_3^2/a_2^3 \qquad (\mu = 0) \tag{20}$$

where a_v are the moments about origin of the standardised variate.

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According to a theorem stated and proved by Cramér [2,p.352] for the case of a function $H(m_v, m_w)$ of two central moments, we have, denoting by H_0, H_1 and H_2 the values assumed by the function H and its first order partial derivatives in the point $m_v = \mu_v$, $m_w = \mu_w$, the mean and the variance of H are given by

$$E(H) = H_0 + O(1/n) \qquad (21)$$

$$\text{Var}(H) = H_1^2 \cdot \text{Var}(m_v) + 2H_1 H_2 \cdot \text{Cov}(m_v, m_w) + H_2^2 \cdot \text{Var}(m_w) + O(1/n^{3/2})$$

Since these formulae are valid for any mean m of the population, including $m = 0$, they are valid also for the moments a_v and r_v.

4.1 Mean and variance of the estimator R_{12} —

From equ.(20) we thus have

$$H(a_1, a_2) = a_2/a_1^2 \qquad (\mu = 0) \qquad (22)$$

and

$$H_0 = g_2/g_1^2 \; ; \quad H_1 = -2g_2/\beta\,g_1^3 \; ; \quad H_2 = 1/\beta^2 g_1^2 \qquad (23)$$

and by equ.(17)

$$n \cdot \text{Var}(r_1) = \beta^2(g_2 - g_1^2) \; ; \quad n \cdot \text{Cov}(r_1, r_2) = \beta^3(g_3 - g_1 g_2) \; ;$$

$$n \cdot \text{Var}(r_2) = \beta^4(g_4 - g_2^2) \qquad (24)$$

which after some calculations results in

$$E(R_{12}) = g_2/g_1^2 \qquad (25)$$

and

$$n \cdot \text{Var}(R_{12}) = (4 g_2^3 + g_1^2 g_4 - 4 g_1 g_2 g_3 - \kappa \; g_2^2)/g_1^6 \qquad (26)$$

If now a value R_{12} has been computed from a random sample, the estimated value α_{12}^+ of the parameter α is obtained by equating the actual value R_{12} to the expected value $E(R_{12})$.

Hence

$$R_{12} = g_2/g_1^2 = f(\alpha_{12}^+) \qquad (27)$$

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Some values of the function $f(\alpha)$ are presented in Table 1.

Since $\text{Var}(R_{12}) \longrightarrow 0$ as $n \longrightarrow \infty$ and $R_{12} \longrightarrow f(\alpha_0)$

where α_0 is the true value of the parameter, it follows that the value a_{12} computed from equ.(27) is a consistent estimate of α. Further, that the tangent curve at α_0 can be substituted for the function $f(\alpha_{12}^+)$.

Hence

$$f(\alpha_0) + (\alpha_{12}^+ - \alpha_0) \cdot f'(\alpha_0) = R_{12} \tag{28}$$

and

$$\text{Var}(\alpha_{12}^+) = \text{Var}(R_{12})/[f'(\alpha_0)]^2 \tag{29}$$

From equ.(14) of SR 7

$$d g_v / d\alpha = v \cdot g_v \, \mathcal{Y}(v\alpha) \tag{30}$$

where $\mathcal{Y}(\alpha) = $ the digamma function, which is tabulated, and

$$f'(\alpha) = d(g_2/g_1^2)/d\alpha = 2 g_2 [\,\mathcal{Y}(2\alpha) - \mathcal{Y}(\alpha)]/g_1^2 \tag{31}$$

The asymptotic efficiency thus becomes

$$n.\text{AsyEff}(\alpha_{12}^+) = [g_2/g_1^2 + g_4/4g_2^2 - g_3/g_1 g_2 - 1/4]/[\,\mathcal{Y}(2\alpha) - \mathcal{Y}(\alpha)]^2 \tag{32}$$

for $\alpha \geq 0.5$.

4.2 Means and variances of the estimators R_{13} and R_{23} —

Proceeding in the same way as above the asymptotic efficiencies are found to be

$$n.\text{AsyEff}(\alpha_{13}^+) = [g_2/g_1^2 + g_6/9g_3^2 - 2g_4/3g_1 g_3 - 4/9]/[\,\mathcal{Y}(3\alpha) - \mathcal{Y}(\alpha)]^2 \tag{33}$$

and

$$n.\text{AsyEff}(\alpha_{23}^+) = [g_4/94g_2^2 + g_6/9g_3^2 - g_5/3g_2 g_3 - 1/36]/[\,\mathcal{Y}(3\alpha) - \mathcal{Y}(2\alpha)]^2 \tag{34}$$

Values of these variances are presented in Table 2.

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4.3 Mean and variance of the estimator M_{23} –

In order to compare the preceeding estimators with the classical estimator based on central moments, the expected value and the asymptotic variance of M_{23} have been computed for various values of α according to formulae presented by Cramér [2,p.357], e.g.,

using the notations

$$k_v = \mu_v / \alpha^v \beta^v \qquad (35)$$

which values are presented in Table 3 of SR7 for $v = 2 \div 6$, we have

$$E(M_{23}) = k_3 / k_2^{3/2} \qquad (36)$$

and

$$n \cdot Var(M_{23}) = k_6 / k_2^3 + 2.25\, k_3^2 k_4 / k_2^5 + 8.75\, k_3^2 / k_2^3 + 9.0 +$$

$$- 3k_3 k_5 / k_2^4 - 6 k_4 / k_2^2 \qquad (37)$$

Some values of $E(M_{23})$ are presented in Table 1, and of $Var(M_{23})$ in Table 2.

4.4 Efficiencies of the estimators

The lower limit attainable of the variances of the estimate of α, denoted by $Var\,\hat{\alpha}$, is given in SR4 for various alternatives. Some values for the case that all three parameters are unknown are presented in Table 2.

The efficiency, as defined by R.A. Fisher, of any estimate is obtained by dividing $Var\,\hat{\alpha}$ by the variance of the estimate in question. Some values resulting from this procedure are presented in Table 3, from which is seen that all the estimators R are more efficient than M_{23} and that R_{12} is the most efficient of them all. It should be noted that this conclusion is valid for $\alpha \geq 0.5$ only. For $\alpha < 0.5$ the estimator M_{23} is more efficient than the estimator R.

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5. Small-sample properties of the estimators R_{12} and M_{23}

5.1 Monte-Carlo study of the distributions of the estimators

An extensive Monte-Carlo study has been performed in order to determine the sampling distributions of the two estimators. By use of an IBM 7090 computer 10,000 random samples for each combination of α_o = 0.001, 0.01, 0.1, 0.3, 0.5, 0.7, 0.9, 1.0, 1.5, 2.0, 5.0 and 10.0 and n = 3, 4, 5, 10 and 20 have been produced, from which the values of R_{12} and M_{23} have been computed and classified in 40 classes. The means and the standard deviations of the ten thousand values of the sixty combinations have at the same time been determined. The values of R_{12} are presented in Tables 4, 5 and 6 for n = 5, 10 and 20, and those of M_{23} in Tables 7, 8 and 9 for n = 5, 10 and 20.

Since the moments m_v and r_v tend to zero as α_o tends to zero, the estimators tend to the ratio 0/0. For this reason, the data corresponding to α_o = 0.001 have been omitted as being somewhat unreliable, and so also the data for α_o = 5.0 and 10.0 as being of less practical interest.

The cumulative distributions of R_{12} and M_{23} have been plotted as demonstrated in Figs.1 and 2, respectively.

5.2 Means, medians and modes of the estimators

The Monte-Carlo determined means $\bar{R}_{12}$ and $\bar{M}_{23}$, computed from the 10,000 values, are presented in Tables 10 and 11, including also the values for α_o = 5.0 and 10.0.

It was found that the statistics $\bar{R}_{12}$, $\check{R}_{12}$ and $\tilde{R}_{12}$ and corresponding statistics with regard to M_{23} could with excellent approximation be expressed as polynomials of the second degree in α_o. Fitted values and corresponding equations are presented in Tables 12 and 13.

The medians $\check{R}_{12}$ and the modes $\tilde{R}_{12}$ have been graphically evaluated from the distribution curves in Fig.1 and the corresponding curves for n = 5 and n = 10. These values are also included in Table 12.

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5.3 Variances of the estimators R_{12} and M_{23}

The Monte-Carlo determined variances $n.Var(R_{12})$ and $n.Var(M_{23})$, obtained by squaring the standard deviations and multiplying the squares by n, are presented in Tables 14 and 15.

5.4 Variances of the estimates α_{12}^+ and α_{23}^{x}

On the condition that the estimate α_{12}^+ is obtained by equating observed values of R_{12} to the expected value of R_{12}, as indicated by equ.(27), the variance of α_{12}^+ is computed from the variance of R_{12} by use of equ.(29). This procedure is strictly correct for $n=\infty$, but an error will be introduced for finite sample sizes. Nevertheless, this method will be applied, and the result will be used as a measure of the variance.

From the equations in Tables 12 and 13 it follows that for

$$
\begin{aligned}
n = 5 \qquad & d\bar{R}_{12}/d\alpha = 0.2835 + 0.4280\,\alpha \\
n = 10 \qquad & d\bar{R}_{12}/d\alpha = 0.2321 + 0.9324\,\alpha \\
n = 20 \qquad & d\bar{R}_{12}/d\alpha = 0.1711 + 1.2724\,\alpha
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
n = 5 \qquad & d\bar{M}_{23}/d\alpha = 1.050 - 0.480\,\alpha \\
n = 10 \qquad & d\bar{M}_{23}/d\alpha = 2.094 - 1.152\,\alpha \\
n = 20 \qquad & d\bar{M}_{23}/d\alpha = 2.830 - 1.552\,\alpha
\end{aligned}
\tag{39}
$$

By use of these formulae and equ.(29) the variances of α_{12}^+ and α_{23}^{x} have been computed. The results are presented in Table 16. It is of interest to note that for $\alpha \geq 0.5$ the variance of α_{23}^{x} is always larger than that of α_{12}^+.

It is obvious that this method of estimation will provide estimates with negative bias of α_{12}^+ and positive bias of α_{23}^{x} but these biases will tend to zero with n. However, if a large set of samples are used for computing R_{12} and the mean of the values R_{12} is introduced into eq.(27), an unbiased estimate will result.

The above-mentioned method of estimation is not the only one possible. It would be quite reasonable to equate the actual value of R_{12} to the median $\check{R}_{12}$. The advantage of doing so would

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be that a correct median estimate will be obtained, that is, there would be 50% probability of having a value larger than and 50% probability of having a value smaller than the true value of α. Most likely, the variance of this estimate will be a little larger, but the bias a little smaller than those of the preceding method.

A third alternative of estimation would be to equate the actual value of R_{12} to the mode $\tilde{R}_{12}$. Such an estimate could be called the most probable estimate. To illustrate this statement by an example, suppose that from a sample of size $n = 20$ a value $R_{12} = 1.45$ has been computed. Knowing nothing about the true value of α, it may be assumed that there is the same probability for all values of α between 0 and 1. (In many cases these will, for logical reasons, be the limits). Then it will be found from Table 6 that, the total number of R_{12} being 7422, the probabilities of a value 1.45 coming from a population with the shape parameter α_o will be for

$$\alpha_o = 0.1, \quad 0.3, \quad 0.5, \; 0.7, \quad 0.9$$

$$P = 5.7 \quad 16.8 \quad 37.1 \; 25.6 \quad 7.9 \; \%$$

The confidence limits of this estimate are easily set. Very likely, the variance of this estimate will be somewhat larger than those of the two preceding methods, but possibly the bias will be the least one.

It should be noted that the differences between the three methods will disappear with increasing sample size, since all the distributions are tending to the same normal distribution.

5.5 Efficiency of the estimates

In order to obtain the efficiency of the various estimates, the variance $\text{Var}\,\hat{\alpha}$ has to be divided by the respective variances. The result thus reached is presented in Table 17. It should be observed that the indicated values of the lower limit $n.\text{Var}\,\hat{\alpha}$ is strictly valid for large sample sizes only, so another approximation has thereby been introduced. However, it is believed that the values will be sufficiently accurate for a comparison between the estimates, even if they do not give the real efficiencies, which may be better than indicated in the table.

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6. Conclusions

An examination of the asymptotic properties of four estimators, three of them based on moments about the smallest sample value and one on central moments, has proved that for $\alpha \geq 0.5$ the first type of estimators is much more efficient than the second type, and that the estimator composed of the first and the second moment about smallest sample value is by far the most efficient one.

Encouraged by this result, an extensive Monte-Carlo study has been performed with the result that the same conclusion applies also to moderate and small sample sizes.

The computational work, when using any one of the moment estimators is considerably less than that required by the Maximum Likelihood Method, which, however, has the advantage of being more efficient.

7. References

1. Lieblein, J.,"On moments of order statistics from the Weibull distribution." Ann.Math.Statist. 26,1955,230-233.

2. Cramèr, H.,Mathematical Methods of Statistics. Princeton Mathematical Series No,9, 1946.

Table 1. Expected values of R_{12} and M_{23} for $n = \infty$

α	$E(R_{12})$	$E(M_{23})$	α	$E(R_{12})$	$E(M_{23})$
0.00	1.0000	-1.1396	0.60	1.3801	0.8960
0.01	1.0002	-1.0809	0.70	1.5045	1.1604
0.10	1.0145	-0.6376	0.80	1.6480	1.4295
0.20	1.0524	-0.2541	0.90	1.8124	1.7080
0.30	1.1093	0.0687	1.00	2.0000	2.0000
0.40	1.1831	0.3586	1.50	3.3953	3.8023
0.50	1.2732	0.6311	2.00	6.0000	6.6188

Table 2. Asymptotic variances of various estimates

α	$n.\text{Var}\,\alpha_{12}^{+}$	$n.\text{Var}\,\alpha_{13}^{+}$	$n.\text{Var}\,\alpha_{23}^{+}$	$n.\text{Var}\,\alpha_{23}^{\boxtimes}$	$n.\text{Var}\,\hat{\alpha}$
0.5	0.15574	0.16715	0.20545	0.88939	0.15198
0.6	0.23484	0.27000	0.36939	1.37268	0.21865
0.7	0.34454	0.42751	0.65061	2.04212	0.29789
0.8	0.49676	0.66876	1.12759	3.33016	0.38908
0.9	0.70731	1.03759	1.92656	5.17201	0.49242
1.0	1.00000	1.60000	3.25000	8.00000	0.60793

Table 3. Asymptotic efficiency of various estimates

α	$\text{Eff}\,\alpha_{12}^{+}$	$\text{Eff}\,\alpha_{13}^{+}$	$\text{Eff}\,\alpha_{23}^{+}$	$\text{Eff}\,\alpha_{23}^{\boxtimes}$
0.5	97.59	90.92	73.97	17.09
0.6	93.19	81.06	59.25	15.94
0.7	86.46	69.68	45.78	14.59
0.8	78.32	58.18	34.50	11.68
0.9	69.62	47.46	25.56	9.52
1.0	60.79	38.00	18.71	7.60

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Table 4. Class frequencies, means and standard deviations of R_{12} — for sample size $n = 5$

Upper class limit	The true shape parameter α_o equal to								
	.01	.1	.3	.5	.7	.9	1.0	1.5	2.0
1.2	0	0	0	0	0	0	0	0	0
1.3	1616	1263	728	386	228	157	131	55	28
1.4	2641	2493	1891	1327	907	650	529	241	138
1.5	1735	1833	1780	1554	1134	815	729	382	202
1.6	1226	1304	1535	1418	1319	1028	910	467	277
1.7	980	1028	1185	1329	1274	1148	1051	697	442
1.8	609	669	842	1041	1063	1050	1018	675	498
1.9	369	424	566	740	882	876	824	695	468
2.0	252	303	417	550	685	784	811	687	521
2.1	184	209	279	409	547	652	680	644	500
2.2	136	145	239	295	428	518	582	620	533
2.3	74	110	146	269	348	429	476	561	523
2.4	55	61	114	171	258	352	374	558	543
2.5	33	45	75	127	203	308	292	472	532
2.6	27	28	54	78	153	210	297	368	459
2.7	14	22	36	67	120	185	204	349	362
2.8	11	16	25	73	95	136	180	285	326
2.9	7	8	23	34	63	142	144	225	290
3.0	5	9	12	34	62	99	129	210	289
3.1	7	6	9	17	60	93	121	202	231
3.2	4	6	17	22	42	62	95	178	257
3.3	3	4	1	12	28	62	69	168	229
3.4	5	4	8	13	23	55	70	155	207
3.5	2	4	4	9	21	49	57	156	188
3.6	2	2	7	5	12	37	63	124	173
3.7	1	1	2	6	17	24	38	123	195
3.8	0	1	2	5	7	17	31	114	167
3.9	0	0	0	4	7	21	28	97	156
4.0	0	0	1	2	1	15	19	108	145
∞	2	2	2	3	13	26	48	384	1121
R_{12}	1.5174	1.5443	1.6166	1.7075	1.8163	1.9395	2.0051	2.3469	2.6718
S.D.	.26665	.28020	.31659	.36469	.42678	.50075	.54039	.73342	.87908

Table 5. Class frequencies, means and standard deviations of R_{12} for sample size $n = 10$

Upper class limit	The true shape parameter α_0 equal to								
	.01	.1	.3	.5	.7	.9	1.0	1.5	2.0
1.1	0	0	0	0	0	0	0	0	0
1.2	2412	1681	443	81	18	5	4	1	0
1.3	3247	3142	2333	961	301	87	65	7	1
1.4	2146	2401	2547	1955	962	429	252	38	9
1.5	1193	1364	1965	2060	1514	838	639	108	22
1.6	539	753	1196	1717	1630	1095	841	212	62
1.7	266	336	727	1212	1461	1279	1040	351	97
1.8	101	183	358	756	1191	1220	1146	441	156
1.9	51	67	205	513	865	1051	1023	545	231
2.0	21	38	95	295	617	872	920	619	285
2.1	13	17	59	188	458	711	778	614	307
2.2	7	10	35	87	308	545	656	637	356
2.3	2	4	16	56	224	455	530	619	422
2.4	1	2	12	47	128	338	421	562	434
2.5	1	1	4	29	92	243	357	497	397
2.6	0	1	2	17	69	200	257	549	414
2.7	0	0	2	8	43	134	225	404	403
2.8	0	0	0	6	31	102	152	406	400
2.9	0	0	0	6	21	84	127	371	361
3.0	0	0	0	1	23	66	115	329	388
3.1	0	0	0	2	11	60	77	285	338
3.2	0	0	0	0	10	43	71	275	349
3.3	0	0	1	0	8	22	53	252	342
3.4	0	0	0	0	7	29	45	206	273
3.5	0	0	0	0	2	11	46	184	290
3.6	0	0	0	0	1	21	22	151	235
3.7	0	0	0	0	0	13	25	128	248
3.8	0	0	0	0	2	15	16	121	217
3.9	0	0	0	0	0	7	18	119	202
4.0	0	0	0	0	0	4	20	89	192
∞	0	0	0	0	3	21	59	880	2569
$\bar{R}_{12}$	1.3108	1.3399	1.4245	1.5426	1.6993	1.8959	2.0083	2.6752	3.4156
S.D.	.14558	.15588	.18590	.23389	.31210	.42749	49847	.93100	1.36031

Upper class limit	The true shape parameter α_0 equal to								
	.01	.1	.3	.5	.7	.9	1.0	1.5	2.0
1.0	0	0	0	0	0	0	0	0	0
1.1	799	286	5	0	0	0	0	0	0
1.2	4947	4191	1297	109	3	0	0	0	0
1.3	3009	3624	3857	1302	165	15	6	0	0
1.4	947	1356	2938	2993	921	166	69	1	0
1.5	240	420	1245	2753	1900	588	276	7	0
1.6	45	93	454	1562	2302	1137	684	25	3
1.7	11	24	147	733	1806	1534	1053	90	4
1.8	1	4	38	329	1244	1535	1274	156	12
1.9	0	1	14	138	701	1370	1320	259	43
2.0	1	1	2	42	398	1058	1191	421	48
2.1	0	0	1	24	289	793	955	479	80
2.2	0	0	1	7	124	534	799	528	142
2.3	0	0	1	4	77	376	582	622	172
2.4	0	0	0	0	22	268	453	633	233
2.5	0	0	0	1	17	194	324	659	265
2.6	0	0	0	2	7	135	238	561	294
2.7	0	0	0	1	12	96	205	581	290
2.8	0	0	0	0	5	59	136	547	354
2.9	0	0	0	0	2	49	100	460	365
3.0	0	0	0	0	3	27	90	402	360
3.1	0	0	0	0	1	17	66	399	378
3.2	0	0	0	0	0	8	45	332	373
3.3	0	0	0	0	1	9	26	300	375
3.4	0	0	0	0	0	8	25	269	362
3.5	0	0	0	0	0	2	19	238	332
3.6	0	0	0	0	0	4	16	210	289
3.7	0	0	0	0	0	5	6	193	315
3.8	0	0	0	0	0	3	4	178	294
3.9	0	0	0	0	1	0	7	121	268
4.0	0	0	0	0	0	4	6	142	241
∞	0	0	0	0	0	6	25	1198	4108
$\bar{R}_{12}$	1.1996	1.2274	1.3133	1.4411	1.6211	1.8605	2.0038	2.9448	4.1628
S.D.	.08691	.09347	.11207	.14486	.20908	.30129	.39863	.99530	1.80312

Table 7. Class frequencies, means and standard deviations of M_{23} for sample size $n = 5$

Upper class limit	The true shape parameter α_o equal to								
	.01	.1	.3	.5	.7	.9	1.0	1.5	2.0
– 2.0	16	0	0	0	0	0	0	0	0
– 1.8	1	0	0	0	0	0	0	0	0
– 1.6	5	0	0	0	0	0	0	0	0
– 1.4	124	85	44	20	15	11	10	4	2
– 1.2	525	411	179	108	61	34	26	14	10
– 1.0	615	550	348	176	100	72	58	25	11
– 0.8	740	639	392	288	176	114	96	33	21
– 0.6	837	830	595	366	269	182	150	78	31
– 0.4	998	1020	875	637	453	320	264	148	82
– 0.2	1325	1316	1199	986	780	638	595	326	234
0.0	1303	1306	1265	1104	889	723	639	412	263
0.2	911	1105	1273	1292	1141	944	867	570	384
0.4	891	1015	1226	1444	1479	1397	1349	1034	739
0.6	557	642	945	1066	1219	1309	1277	1260	1260
0.8	430	422	596	812	973	1044	1079	923	852
1.0	263	306	445	677	789	913	978	1031	842
1.2	197	206	346	516	763	894	929	1200	1105
1.4	127	121	213	404	683	985	1110	1447	1649
1.6	56	26	59	104	210	420	573	1495	2515
1.8	22	0	0	0	0	0	0	0	0
2.0	10	0	0	0	0	0	0	0	0
∞	47	0	0	0	0	0	0	0	0
$\bar{M}_{23}$	-.1948	-.1732	.0148	.1924	.3486	.4812	.5392	.7630	.9099
S.D.	1.07705	.61352	.60428	.60034	.59812	.59401	.59089	.56718	.53784

Table 8. Class frequencies, means and standard deviations of M_{23} for sample size $n = 10$

Upper class limit	The true shape parameter α_o equal to								
	.01	.1	.3	.5	.7	.9	1.0	1.5	2.0
− 2.0	123	34	3	0	0	0	0	0	0
− 1.8	131	75	4	1	0	0	0	0	0
− 1.6	235	106	9	2	1	0	0	0	0
− 1.4	334	190	37	6	2	1	0	0	0
− 1.2	514	320	71	10	2	1	1	0	0
− 1.0	775	561	165	32	10	5	2	0	0
− 0.8	947	832	303	79	20	5	6	0	0
− 0.6	1090	1055	554	201	68	28	13	5	0
− 0.4	1278	1249	890	386	161	62	46	9	0
− 0.2	1288	1378	1251	691	321	162	113	25	13
0.0	1201	1292	1503	1105	617	329	246	60	19
0.2	819	1111	1466	1399	1002	637	500	158	60
0.4	573	791	1295	1501	1259	927	796	335	139
0.6	342	487	997	1354	1434	1219	1053	553	314
0.8	197	288	676	1176	1348	1342	1269	857	498
1.0	81	115	386	853	1185	1284	1294	1008	774
1.2	35	62	185	503	912	1151	1221	1089	877
1.4	23	33	112	316	647	916	1011	1226	1057
1.6	10	10	47	184	391	665	767	1104	1167
1.8	2	8	28	117	278	461	571	789	938
2.0	0	2	11	46	177	341	433	786	809
2.2	2	0	6	28	107	255	330	696	851
2.4	0	1	0	8	48	148	218	625	935
2.6	0	0	1	2	9	58	102	561	1079
2.8	0	0	0	0	1	3	8	114	470
∞	0	0	0	0	0	0	0	0	0
$\bar{M}_{23}$	−.4976	−.3324	.0363	.3670	.6478	.8821	.9840	1.3768	1.6356
S.D.	.63159	.59231	.54587	.54962	.57581	.60266	.61371	.64178	.63899

Table 9. Class frequencies, means and standard deviations of M_{23} for sample size $n = 20$

Upper class limit	The true shape parameter α_0 equal to								
	.01	.1	.3	.5	.7	.9	1.0	1.5	2.0
- 2.2	160	21	0	0	0	0	0	0	0
- 2.0	119	35	0	0	0	0	0	0	0
- 1.8	185	52	0	0	0	0	0	0	0
- 1.6	297	89	0	0	0	0	0	0	0
- 1.4	447	203	5	0	0	0	0	0	0
- 1.2	667	351	14	0	0	0	0	0	0
- 1.0	948	570	50	0	0	0	0	0	0
- 0.8	1266	999	134	5	0	0	0	0	0
- 0.6	1474	1322	340	34	0	0	0	0	0
- 0.4	1528	1649	808	94	16	0	0	0	0
- 0.2	1278	1657	1399	317	48	10	3	0	0
0.0	862	1401	1905	796	186	52	28	0	0
0.2	445	853	1913	1485	559	150	93	5	0
0.4	200	479	1528	1860	988	453	273	34	3
0.6	85	192	982	1762	1564	823	584	113	25
0.8	31	87	510	1464	1674	1284	1013	254	73
1.0	7	32	242	964	1494	1485	1366	491	180
1.2	1	7	103	559	1234	1424	1374	819	338
1.4	0	1	44	333	812	1260	1289	972	593
1.6	0	0	12	167	547	927	1106	1110	771
1.8	0	0	10	90	368	663	824	1112	888
2.0	0	0	1	35	207	492	589	1034	928
2.2	0	0	0	18	130	359	481	888	935
2.4	0	0	0	10	86	224	335	693	973
2.6	0	0	0	6	44	140	230	595	820
2.8	0	0	0	1	22	116	135	451	640
3.0	0	0	0	0	13	64	128	400	554
3.2	0	0	0	0	5	35	70	334	499
3.4	0	0	0	0	3	26	43	253	448
3.6	0	0	0	0	0	8	26	178	471
3.8	0	0	0	0	0	5	9	173	392
4.0	0	0	0	0	0	0	1	78	321
4.2	0	0	0	0	0	0	0	13	148
∞	0	0	0	0	0	0	0	0	0
$\overline{M}_{23}$	-.7315	-.4630	.0471	.4830	.8569	1.1807	1.3263	1.9187	2.3368
S.D.	.57480	.49993	.41990	.45010	.52099	.59586	.63010	.75271	.80651

Table 10. Monte-Carlo determined values of $\bar{R}_{12}$ —

α_o	n = 3	n = 4	n = 5	n = 10	n = 20
.01	1.7756	1.6165	1.5174	1.3108	1.1996
.1	1.7919	1.6402	1.5443	1.3399	1.2274
.3	1.8320	1.7022	1.6166	1.4245	1.3133
.5	1.8773	1.7766	1.7075	1.5426	1.4411
.7	1.9261	1.8617	1.8163	1.6993	1.6211
.9	1.9767	1.9545	1.9395	1.8959	1.8605
1.0	2.0021	2.0026	2.0051	2.0083	2.0038
1.5	2.1238	2.2441	2.3469	2.6752	2.9448
2.0	2.2299	2.4643	2.6718	3.4156	4.1628
5.0	2.5807	3.2065	3.8012	6.4644	10.6992
10.0	2.7700	3.5861	4.3714	8.1353	14.9989

Table 11. Monte-Carlo determined values of $\bar{M}_{23}$ —

α_o	n = 3	n = 4	n = 5	n = 10	n = 20
.01	–	-.3138	-.1948	-.4976	-.7315
.1	-.0500	-.1171	-.1732	-.3324	-.4630
.3	.0076	.0143	.0148	.0363	.0471
.5	.0708	.1395	.1924	.3670	.4830
.7	.1285	.2516	.3486	.6478	.8569
.9	.1798	.3487	.4812	.8821	1.1807
1.0	.2030	.3918	.5392	.9840	1.3263
1.5	.2976	.5614	.7630	1.3768	1.9187
2.0	.3650	.6760	.9099	1.6356	2.3368
5.0	.5408	.9441	1.2413	2.2215	3.3489
10.0	.6200	1.0494	1.3684	2.4426	3.7436

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Table 12. The statistics $\bar{R}$, $\breve{R}$ and $\tilde{R}$ as functions of α_o and n

α_o	n = 5			n = 10			n = 20		
	$\bar{R}$	$\breve{R}$	$\tilde{R}$	$\bar{R}$	$\breve{R}$	$\tilde{R}$	$\bar{R}$	$\breve{R}$	$\tilde{R}$
.0	1.5123	1.4300	1.3800	1.3100	1.2780	1.2250	1.1965	1.1800	1.1630
.1	1.5428	1.4635	1.4070	1.3379	1.3082	1.2550	1.2200	1.2079	1.1898
.2	1.5776	1.4995	1.4360	1.3751	1.3454	1.2900	1.2562	1.2461	1.2252
.3	1.6166	1.5379	1.4670	1.4215	1.3898	1.3300	1.3051	1.2947	1.2694
.4	1.6599	1.5787	1.5000	1.4774	1.4414	1.3750	1.3667	1.3537	1.3224
.5	1.7075	1.6220	1.5350	1.5426	1.5000	1.4250	1.4411	1.4230	1.3840
.6	1.7594	1.6677	1.5720	1.6171	1.5658	1.4800	1.5282	1.5027	1.4644
.7	1.8156	1.7159	1.6110	1.7009	1.6386	1.5400	1.6280	1.5927	1.5334
.8	1.8760	1.7665	1.6520	1.7940	1.7186	1.6050	1.7406	1.6931	1.6212
.9	1.9408	1.8195	1.6950	1.8965	1.8058	1.6750	1.8658	1.8039	1.7178
1.0	2.0098	1.8750	1.7400	2.0083	1.9000	1.7500	2.0038	1.9250	1.8230

$$\underset{=========}{n = 5}$$

$$\bar{R} = 1.5123 + 0.2835\,\alpha + 0.2140\,\alpha^2 \quad ; \quad \alpha = -0.6624 + 2.162\,\sqrt{\bar{R} - 1.4184}$$

$$\breve{R} = 1.4300 + 0.3230\,\alpha + 0.1220\,\alpha^2 \quad ; \quad \alpha = -1.3238 + 2.863\,\sqrt{\breve{R} - 1.2162}$$

$$\tilde{R} = 1.3800 + 0.2600\,\alpha + 0.1000\,\alpha^2 \quad ; \quad \alpha = -1.3000 + 3.162\,\sqrt{\tilde{R} - 1.2110}$$

$$\underset{==========}{n = 10}$$

$$\bar{R} = 1.3100 + 0.2321\,\alpha + 0.4662\,\alpha^2 \quad ; \quad \alpha = -0.2489 + 1.465\,\sqrt{\bar{R} - 1.2811}$$

$$\breve{R} = 1.2780 + 0.2660\,\alpha + 0.3560\,\alpha^2 \quad ; \quad \alpha = -0.3736 + 1.676\,\sqrt{\breve{R} - 1.2283}$$

$$\tilde{R} = 1.2250 + 0.2750\,\alpha + 0.2500\,\alpha^2 \quad ; \quad \alpha = -0.5500 + 2.000\,\sqrt{\tilde{R} - 1.1494}$$

$$\underset{==========}{n = 20}$$

$$\bar{R} = 1.1965 + 0.1711\,\alpha + 0.6362\,\alpha^2 \quad ; \quad \alpha = -0.1345 + 1.254\,\sqrt{\bar{R} - 1.1850}$$

$$\breve{R} = 1.1800 + 0.2270\,\alpha + 0.5180\,\alpha^2 \quad ; \quad \alpha = -0.2191 + 1.389\,\sqrt{\breve{R} - 1.1551}$$

$$\tilde{R} = 1.1630 + 0.2240\,\alpha + 0.4360\,\alpha^2 \quad ; \quad \alpha = -0.256\cap \qquad \sqrt{\tilde{R} - 1.1342}$$

Table 13. The means $\overline{M}_{23}$ as functions of α_o and n

α_o	Values of $\overline{M}_{23}$ for			
	$n = 5$	$n = 10$	$n = 20$	$n = \infty$
.0	-0.2766	-0.5360	-0.7383	-1.1396
.1	-0.1732	-0.3324	-0.4631	-0.6376
.2	-0.0746	-0.1402	-0.2033	-0.2541
.3	0.0192	0.0404	0.0409	0.0687
.4	0.1082	0.2094	0.2695	0.3586
.5	0.1924	0.3670	0.4827	0.6311
.6	0.2718	0.5130	0.6803	0.8960
.7	0.3464	0.6476	0.8625	1.1604
.8	0.4162	0.7706	1.0291	1.4295
.9	0.4812	0.8820	1.1801	1.7080
1.0	0.5414	0.9820	1.3157	2.0000

$$\underline{n = 5}$$

$$\overline{M}_{23} = -0.2766 + 1.050\,\alpha - 0.240\,\alpha^2; \quad \alpha = 2.2042 - 2.041\sqrt{0.8894 - \overline{M}_{23}}$$

$$\underline{n = 10}$$

$$\overline{M}_{23} = -0.5360 + 2.094\,\alpha - 0.576\,\alpha^2; \quad \alpha = 1.8177 - 1.318\sqrt{1.3671 - \overline{M}_{23}}$$

$$\underline{n = 20}$$

$$\overline{M}_{23} = -0.7383 + 2.830\,\alpha - 0.776\,\alpha^2; \quad \alpha = 1.8235 - 1.135\sqrt{1.8420 - \overline{M}_{23}}$$

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Table 14. Monte-Carlo determined values of $n.Var(R_{12})$

α_0	n = 3	n = 4	n = 5	n = 10	n = 20
.01	0.3870	0.3881	0.3555	0.2112	0.1510
.1	0.4016	0.4197	0.3926	0.2430	0.1748
.3	0.4373	0.5061	0.5012	0.3456	0.2512
.5	0.4780	0.6209	0.6650	0.5470	0.4196
.7	0.5239	0.7716	0.9107	0.9741	0.8742
.9	0.5730	0.9580	1.2538	1.8275	2.0646
1.0	0.5979	1.0617	1.4601	2.4847	3.1782
1.5	0.7113	1.6115	2.6895	8.6676	19.8124
2.0	0.7896	2.0745	3.8639	18.5044	65.0248
5.0	0.7760	2.4740	5.4802	50.7443	404.3828
10.0	0.5350	1.6978	3.9759	42.5263	406.5855

Table 15. Monte-Carlo determined values of $n.Var(M_{23})$

α_0	n = 3	n = 4	n = 5	n = 10	n = 20
.01	—	—	5.8002	3.9891	6.6079
.1	1.6350	1.3610	1.8820	3.5083	4.9986
.3	0.7443	1.3430	1.8258	2.9797	3.5264
.5	0.7424	1.3346	1.8020	3.0208	4.0518
.7	0.7376	1.3204	1.7888	3.3156	5.4286
.9	0.7280	1.2942	1.7542	3.6320	7.1010
1.0	0.7214	1.2764	1.7458	3.7664	7.9406
1.5	0.6759	1.1592	1.6084	4.1188	11.3314
2.0	0.6216	1.0308	1.4464	4.0831	13.0092
5.0	0.3744	0.5501	0.8106	2.6794	10.6492
10.0	0.2154	0.2948	0.4476	1.5450	6.3194

Table 16. The variances $n.\mathrm{Var}\,\alpha_{12}^{+}$ and $n.\mathrm{Var}\,\alpha_{23}^{\equiv}$ as functions of α_o and n

α_o	n = 5		n = 10		n = 20		n = ∞	
	$n.\mathrm{Var}\,\alpha^{+}$	$n.\mathrm{Var}\,\alpha^{\equiv}$	$n.\mathrm{Var}\,\alpha^{+}$	$n.\mathrm{Var}\,\alpha^{\equiv}$	$n.\mathrm{Var}\,\alpha^{+}$	$n.\mathrm{Var}\,\alpha^{\equiv}$	$n.\mathrm{Var}\,\alpha^{+}$	$n.\mathrm{Var}\,\alpha^{\equiv}$
.01	4.2919	5.3093	3.6241	0.9198	4.4698	0.8342	–	0.8647
.1	3.6870	1.8745	2.2963	0.8960	1.9644	0.6986	–	0.5507
.3	2.9539	2.2243	1.3194	0.9748	0.8220	0.6308	–	0.4305
.5	2.6867	2.7465	1.1218	1.3109	0.6438	0.9604	0.1557	0.8894
.7	2.6786	3.5089	1.2443	1.9999	0.7754	1.7857	0.3445	2.0421
.9	2.8036	4.6192	1.5924	3.2496	1.1916	3.4570	0.7073	5.1720
1.0	2.8843	5.3733	1.8323	4.2445	1.5253	4.8617	1.0000	8.0000

Table 17. The efficiencies $\mathrm{Eff}\,\alpha_{12}^{+}$ and $\mathrm{Eff}\,\alpha_{23}^{\equiv}$ as functions of α_o and n

α_o	n = 5		n = 10		n = 20		n = ∞		
	$\mathrm{Eff}\,\alpha^{+}$	$\mathrm{Eff}\,\alpha^{\equiv}$	$\mathrm{Eff}\,\alpha^{+}$	$\mathrm{Eff}\,\alpha^{\equiv}$	$\mathrm{Eff}\,\alpha^{+}$	$\mathrm{Eff}\,\alpha^{\equiv}$	$\mathrm{Eff}\,\alpha^{+}$	$\mathrm{Eff}\,\alpha^{\equiv}$	$n.\mathrm{Var}\,\hat{\alpha}$
.0	–	–	–	–	–	–	–	53.30	0.4765
.1	10.75	21.14	17.26	44.22	20.17	56.72	–	72.79	0.4008
.2	–	–	–	–	–	–	–	88.34	0.3304
.3	8.99	11.94	20.12	27.24	32.30	42.09	–	61.63	0.2653
.4	–	–	–	–	–	–	–	34.63	0.2058
.5	5.66	5.53	13.55	11.59	23.60	15.82	97.59	17.09	0.1520
.6	–	–	–	–	–	–	93.19	15.94	0.2188
.7	11.12	8.49	23.94	14.89	38.41	16.68	86.46	14.59	0.2979
.8	–	–	–	–	–	–	78.32	11.68	0.3891
.9	17.56	10.66	30.92	15.15	41.32	14.24	69.62	9.52	0.4924
1.0	21.07	11.31	33.17	14.32	39.85	12.50	60.79	7.60	0.6079
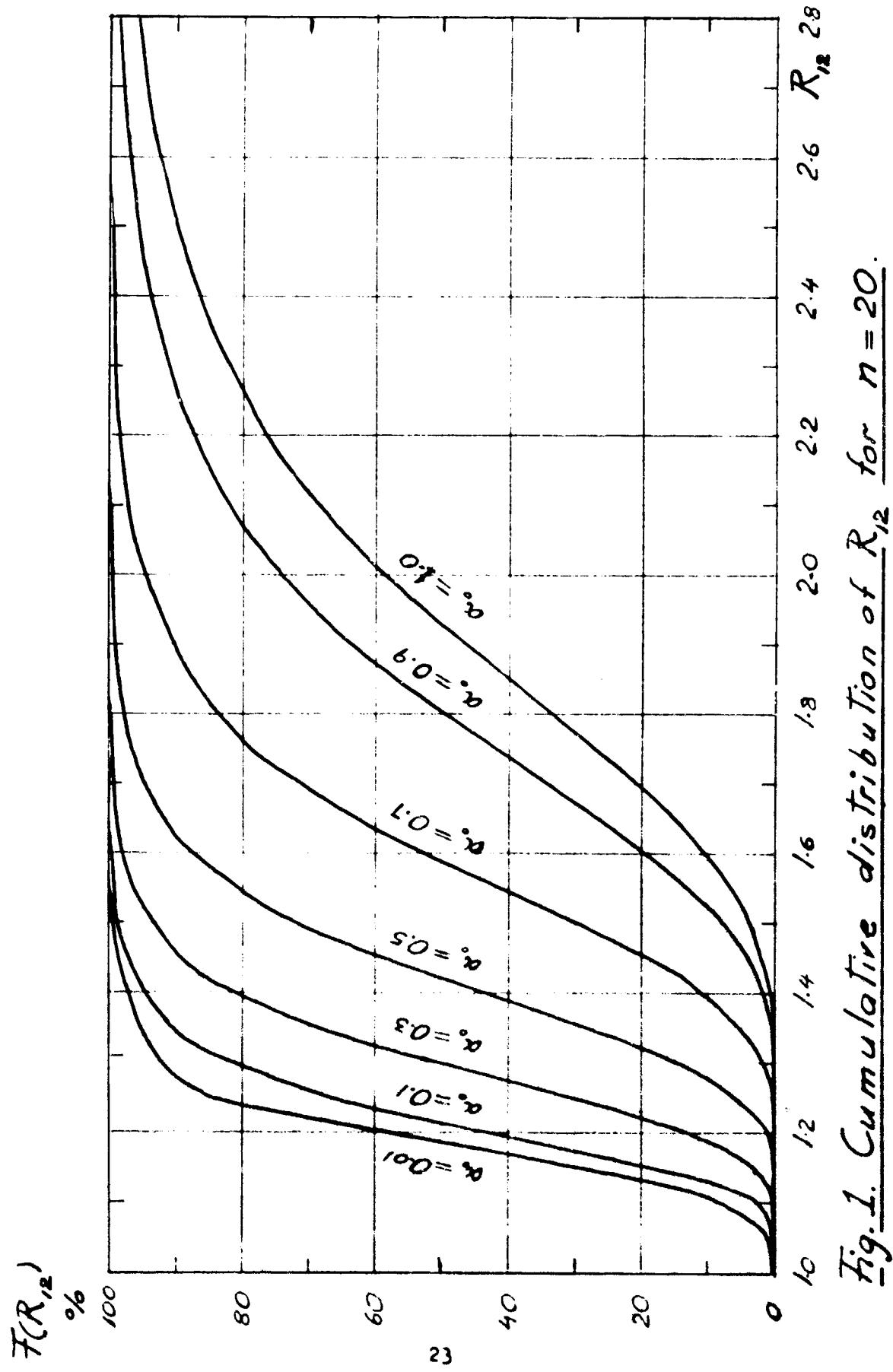
Fig. 1. Cumulative distribution of R_{12} for $n=20$.

$F(R_{12})$ %

R_{12}

$\alpha_0 = 0.1$

$\alpha_0 = 0.9$

$\alpha_0 = 0.7$

$\alpha_0 = 0.5$

$\alpha_0 = 0.3$

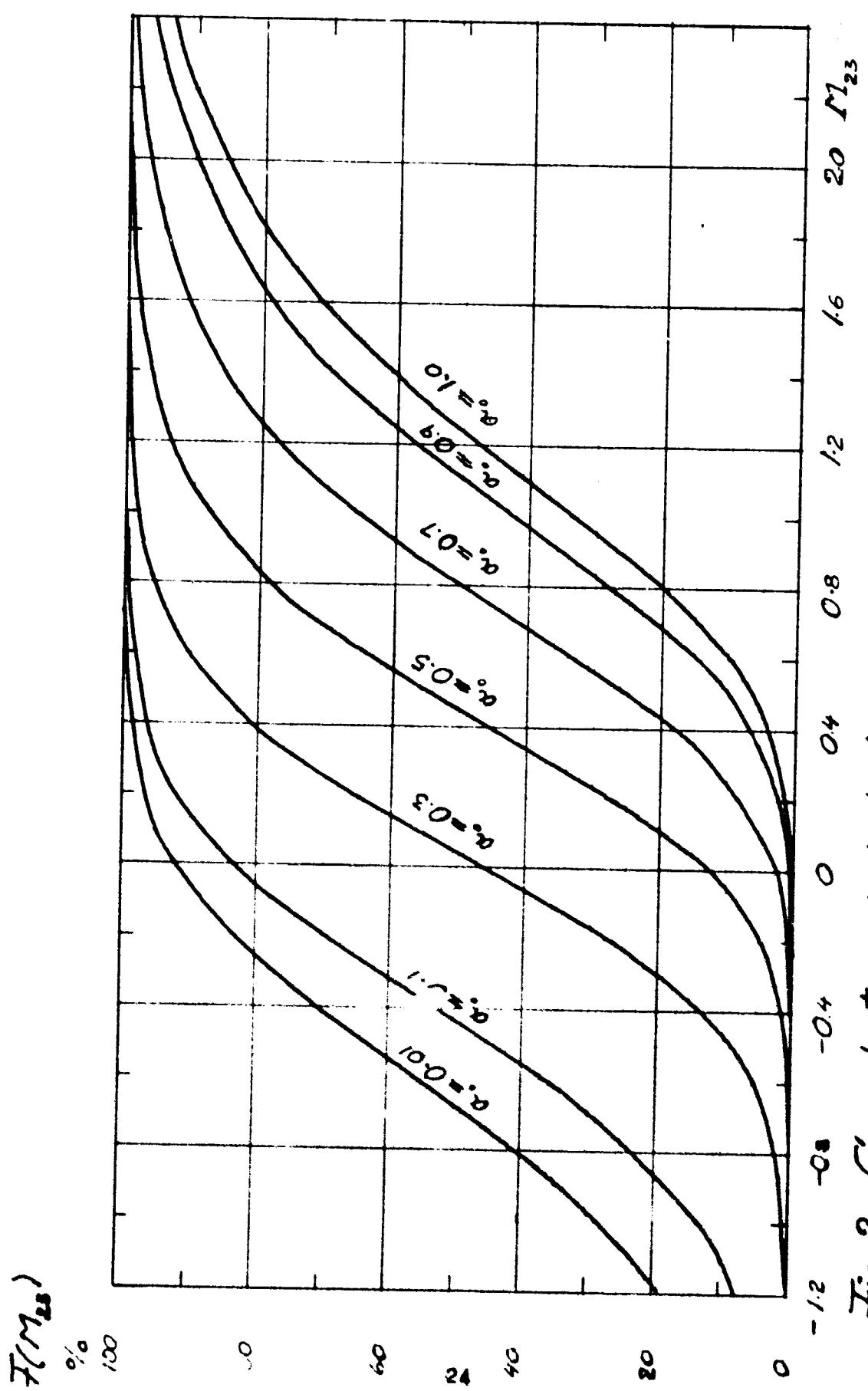$\alpha_0 = 0.1$
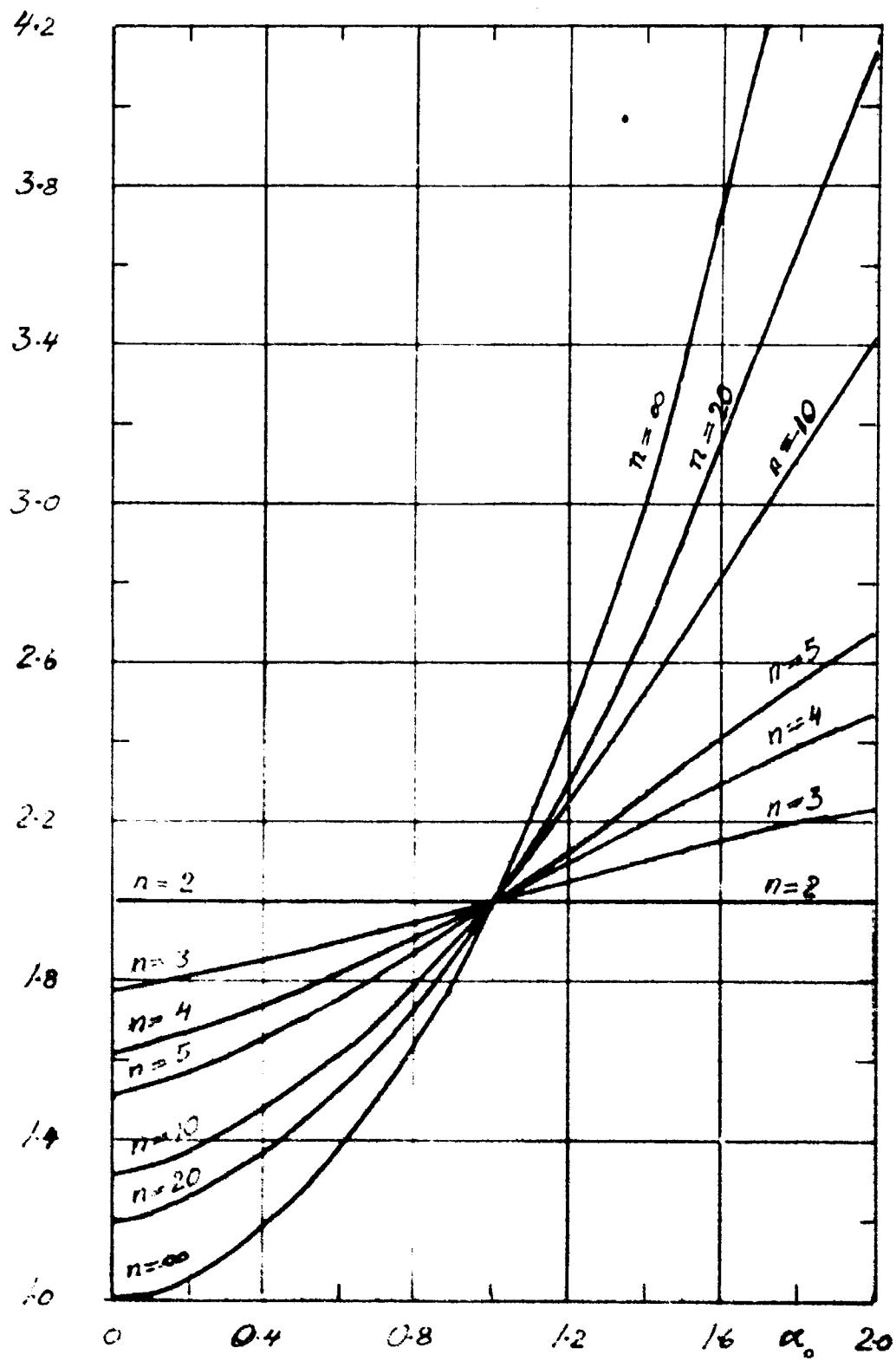
$\alpha_0 = 0.01$

23

Fig. 2. Cumulative distribution of M_{23} for $n = 20$

Fig 3. The mean $\bar{R}_{12}$ as function of α_o and n
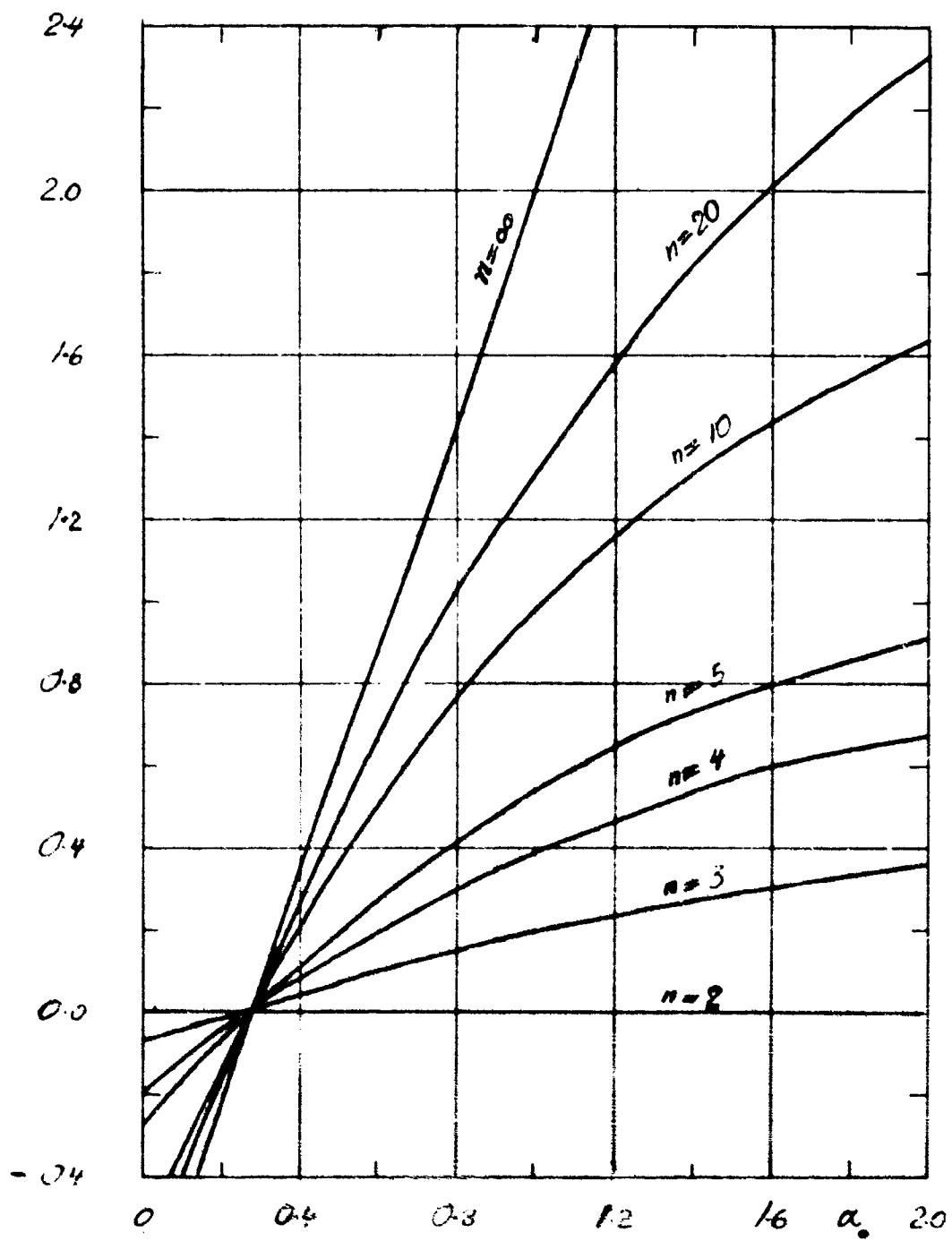
Fig. 4. The mean $\bar{M}_{23}$ as function of α_o and n

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13 ABSTRACT

A new type of moments has been achieved by substituting in the central moments the smallest value of the sample for its mean. The new moments have the same advantage as the central moments of being independent of the location parameter but for certain value of the shape parameter they have less variance and thus are preferable for estimating purposes. The asymptotic properties of four estimators, three of them composed of the new moments and one of them of central moments have been examined. It could be conculded that for the shape parameter $d \geq 0.5$ the estimator, which was composed of the first and second order moments of the new type, was by far the most efficeint one. Small-sample properties of the new-moments estimators have been appraised by use of ex tensive Monte-Carlo studies and it could be stated that the same conclusion applies also to small and moderate sample sizes.

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